Math 210C Lecture 27 Notes

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1 Character Table Calculations and Induced Representations

1.1 Calculating character tables using orthogonality relations

Last time, we proved orthogonality relations for irreducible characters.

Let G be a finite group of order n, let $\chi_1, \dots, \chi_r$ be the irreuducible characters (with corresponding representations V_i with dimension n_i), and let $g_1 \in C_1, \dots, g_r \in C_r$ be representative of the conjugacy classes with orders c_i.

Example 1.1. Let's calculate the character table of D_4. How many 1-dimensional representations does D_4 have? It should be the order of the abelianization of D_4. So $\chi_1, \dots, \chi_4$ should be 1-dmensional represntations, i.e. homomorphisms to $\mathbb{C}$.

Since $|D_4| = 8$, we have $4 \cdot 1 + 2^2 = 6 = |D_4|$, so $n_5 = 2$. Using the orthogonality relations, we can find the missing irreducible representation.

D_4	e	s	r	rs	r^2
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	1	1	-1	-1	1
χ_4	1	-1	-1	1	1
χ_5	2	0	0	0	-2

Example 1.2. Let G be a nonabelian group of order 8. Then $G \cong D_4$ or Q_8. You can check that Q_8 has the same character table as D_4. So the character table of a group does not uniquely determine the group.

Example 1.3. Let's figure out the character table of S_4. S_4 has 5 conjugacy classes. We also know that $\sum_{i=1}^5 n_i^2 = |S_4| = 24$. We know that χ_1 is trivial and χ_2 is the sign character. So we have $1 + 1 + x^2 + y^2 + z^2 = 24$. Then we must have $x = 2$ and $y = z = 3$ (without loss of generality).

There is a normal subgroup $N = \langle (1\,2)(3\,4), (1\,3)(2\,4) \rangle \trianglelefteq S_4$, and $S_4/N \cong S_3$. So we get representations of S_4 by factoring through representations of S_3. This gives χ_3, which corresponds to the 2-dimensional irreducible representation of S_3.

Using the orthogonality relations, we can solve for the last two rows of the character table.

S_4	e	$(1\,2)$	$(1\,2\,3)$	$(1\,2\,3\,4)$	$(1\,2)(3\,4)$
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	2	0	-1	0	2
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	1	-1

Example 1.4. The character table for A_4 is similar, but $A_4^{\text{ab}} \cong \mathbb{Z}/3\mathbb{Z}$, so there should be 3 1-dimensional irreducible representations:

S_4	e	$(1\,2\,3)$	$(1\,2\,3\,4)$	$(1\,2)(3\,4)$
χ_1	1	1	1	1
χ_2	1	ω	ω^2	1
χ_3	1	ω^2	ω	0
χ_4	3	0	0	-1

Here $\omega = e^{2\pi i/3}$.

1.2 Induced representations and Frobenius reciprocity

Let R be a commutative ring, and let G be a group with $H \leq G$.

Definition 1.1. Let A be an $R[G]$-module. The **restriction** $\operatorname{Res}_H^G(A)$ fo A from G to H is A viewed as an $R[H]$-module.

This gives a functor $\operatorname{Res}_H^G : R[G]\text{-Mod} \to R[H]\text{-Mod}$. This is an exact functor. Does it have an adjoint?

Definition 1.2. The **induced module** of an $R[H]$-module B is the $R[G]$-module

$$\operatorname{Ind}_H^G(B) = \operatorname{Hom}_{R[H]}(R[G], B)$$

with action $g \circ \varphi(x) = \varphi(xg)$ for $x \in R[G]$.

This gives a functor $\operatorname{Ind}_H^G : R[H]\text{-Mod} \to R[G]\text{-Mod}$. This is an exact functor, as $R[G]$ is $R[H]$-free (and hence projective).

We also have a functor $B \mapsto R[G] \otimes_{R[H]} B$, which is exact. The action is defined by $g \cdot (x \otimes b) = gx \otimes b$.

Proposition 1.1. *If $H \leq G$ has finite index, then there exist natural isomorphisms $k : \mathrm{Ind}_H^G(B) \to R[G] \otimes_{R[H]} B$ sending $\varphi \mapsto \sum_{\overline{g} \in H \backslash G} g^{-1} \otimes \varphi(g)$.*

Remark 1.1. In group cohomology, $H^1(H, B) \cong H^1(G, \mathrm{Ind}_H^G(B))$ for a $\mathbb{Z}[H]$-module B.

Proposition 1.2. *Let $H \leq G$ have finite index. Then Ind_H^G is left adjoint to Res_H^G.*

Proof. By the tensor-hom adjunction,

$$\begin{aligned}\mathrm{Hom}_{R[G]}(\mathrm{Ind}_H^G(A), B) &\cong \mathrm{Hom}_{R[G]}(R[G] \otimes_{R[H]} A, B)\\ &\cong \mathrm{Hom}_{R[G]}(A, \mathrm{Hom}_{R[G]}(R[G], B))\end{aligned}$$

Evaluate at 1.

$$\cong \mathrm{Hom}_{R[H]}(A, B),$$

where in the last line B is viewed as $\mathrm{Res}_H^G(B)$. □

Definition 1.3. Let W be a finite dimensional H presentation with character ψ. Then $\mathrm{Ind}_H^G)(W)$ is an **induced representation** with $\dim = [G : H] \dim(W)$, and $\mathrm{Ind}_H^G(\psi)$ is an **induced character**, the character of $\mathrm{Ind}_H^G)(W)$.

Example 1.5. If G is finite, $\mathrm{Ind}_H^G(G) = F[G] \otimes_{F[H]} F \cong F[G/H]$. Here, g acts left on G/H by the permutation representation of G on G/H. If F has character ψ_1, then $\mathrm{Ind}(H^G(\psi_1)(g)$ is the number of cosets g_iH fixed by G. Then $gg_iH = g_iH \iff g_i^{-1}gg_i \in H$.

Example 1.6. If $[G : H] < \infty$, then $\mathrm{Ind}_H^G(F[H]) \cong F[G]$. This is because $\mathrm{Ind}_H^G(F[H]) \cong \bigoplus_{j=1}^s \mathrm{Ind}_H^G(W_i)^{m_i}$, where the W_i are the irreducible representations of H. On the other hand, $F[G] \cong \bigoplus_{i=1}^r V_i^{n_i}$. So every irreducible character of G appears in writing the irreducible characters of the induced characters of H as sums of irreducible characters of G.

Theorem 1.1 (Frobenius reciprocity)**.** *If ψ is a $\mathbb{C}$-character of H and χ is a $\mathbb{C}$-character of G, then*

$$\left\langle \mathrm{Ind}_H^G(\psi), \chi \right\rangle_G = \left\langle \psi, \mathrm{Res}_H^G(\chi) \right\rangle_H .$$

Proof. The left hand side is

$$\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}[G]}(\mathrm{Ind}_H^G(U), V)) = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}[H]}(U, \mathrm{Res}_H^G(V)),$$

which is the right hand side. □